Debasish Das
Guest Lecturer
Department of Physics
M. G. M

Electromagnetic Theory

Paper - C-13T

6th Sem

Electromagnetic field ($\vec{E}$ & $\vec{B}$) : Source of this fields are charges & currents.

For classical electrodynamics charge/current distribution are continuo

charge density $\rho = \rho(\vec{r}, t)$ and current density $\vec{j} = \vec{j}(\vec{r}, t)$

Maxwell eqns

① $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

② $\vec{\nabla}\times\vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

③ $\vec{\nabla}\cdot\vec{B} = 0$

④ $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

Q. why these are called eqns of motions?

observation-1

The electromagnetic field is a dynamical system given by the values of E_x, E_y, E_z & B_x, B_y, B_z at every point in space at any instant of time.

From these the spatial derivatives of the field can be calculated. Maxwells eqns gives us the rate of change with time. In this sense it is the eqn of motion.

Obs.-2

The Maxwells eqns contain the continuity eqn for charge ~~distribution~~ conservation. (Hints take divergence of the eqn ②

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0 \vec{\nabla}\cdot\vec{j} + \mu_0\epsilon_0 \vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t} = 0$$

Obs.-3

the eqns ① and ② are inhomogeneous differential eqn & eqn ③ and ④ are homogeneous differential eqn.

Obs. 4

Maxwell's eqn are a set of first order simultaneo ~~Coup~~ coupled ~~eqn.~~ differential eqn. ($\vec{E}$ increases with B & again B with E)

Obs.-5 How many dependent variables are there?

$\Rightarrow$ 6

How many eqn are there?

$\Rightarrow$ 6

eqn ③ is the initialisation of eqn ④

From eqn ③ $\vec{\nabla}\cdot\vec{B} = 0$

From eqn ④ $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{E} + \vec{\nabla}\cdot\frac{\partial \vec{B}}{\partial t} = 0$$

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$$

$$\vec{\nabla}\cdot\vec{B} = \text{Const in time}$$

Similarly, From eqn. (1) $\vec{\nabla}\cdot\vec{E} - \rho/\epsilon_0 = 0$

and From eqn. (2) $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$.

or, $\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} = \mu_0\vec{\nabla}\cdot\vec{j} + \mu_0\epsilon_0\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}$

or, $\vec{\nabla}\cdot\vec{j} + \epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$

or, $-\frac{\partial\rho}{\partial t} + \epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$

or, $\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E} - \rho/\epsilon_0\right) = 0$

$\vec{\nabla}\cdot\vec{E} - \rho/\epsilon_0 =$ Const in time.

the eqn. (1) is the initialisation of eqn. (2)

So, there are Six eqn. of motion.

Thus it is shown that for 6 variables we have 6 eqn.

Electric field in material.

Bound charge density, $\rho_b = -\vec{\nabla}\cdot\vec{P}$ $\qquad \vec{P} \Rightarrow$ Polarisati

Now, $\vec{\nabla}\cdot\vec{E} = \frac{\rho_f + \rho_b}{\epsilon_0}$

$\vec{\nabla}\cdot(\epsilon_0\vec{E}) = \rho_f - \vec{\nabla}\cdot\vec{P}$

or, $\vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$

$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$ —— (1) first Maxwell's eqn. for material medium

$\therefore \vec{D} = \epsilon_0\vec{E} + \vec{P}$ = electric displacement vector (Electric field in material)

for linear isotropic medium, $\vec{P} = \epsilon_0\chi_e\vec{E}$

$\vec{D} = \epsilon_0(1+\chi_e)\vec{E}$ | $\epsilon = \epsilon_0(1+\chi_e)$

$\boxed{\vec{D} = \epsilon\vec{E}}$ | $K = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = dielectric Const in

Magnetic field in material.

Magnetisation current density, $\vec{j}_b = \vec{\nabla}\times\vec{M}$ $\qquad \vec{M} \Rightarrow$ Magnetis.

and Polarisation current density, $\vec{j}_p = \frac{\partial\vec{P}}{\partial t}$

Now, From the eqn. (2) we get

$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \vec{j}_f + \vec{j}_b + \vec{j}_p + \epsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{j}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$

$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{j}_f + \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P})$

$\boxed{\vec{\nabla}\times\vec{H} = \vec{j}_f + \frac{\partial\vec{D}}{\partial t}}$ $\Rightarrow$ 2nd Maxwell eqn. in medium

Note: The homogeneous set of eqns is unchanged.

$\vec{H} = \vec{B}/\mu_0 - \vec{M}$

$\vec{B} = \mu_0(\vec{H}+\vec{M})$

$\vec{D} = \epsilon_0\vec{E} + \vec{P}$, $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ } Constitutive relations

For linear isotropic medium.

$\vec{M} = \chi_m \vec{H}$

$\vec{B} = \mu_0(1+\chi_m)\vec{H}$

$\vec{B} = \mu\vec{H}$

$\mu = \mu_0(1+\chi_m)$

$\mu_r = \frac{\mu}{\mu_0} = 1+\chi_m$ = relative Permeability in the medium.

Maxwell's eqns inside a medium

(1) $\vec{\nabla}\cdot\vec{D} = \rho_f$.

(2) $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$

(3) $\vec{\nabla}\cdot\vec{B} = 0$

(4) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

Helmholtz Theorem :-

A Vector field is Completely specified by the values of its divergence and curl everywhere in space and if the sources producing the field vanishes sufficiently rapidly at infinity.

Suppose we are told that the divergence of a vector function $\vec{F}(r)$ is a specified scalar function $D(r)$

$$\vec{\nabla}\cdot\vec{F} = D(r) \quad \text{--- (1)}$$

and curl of $F(r)$ is a specified vector function $C(r)$.

$$\vec{\nabla}\times\vec{F} = \vec{C}(r) \quad \text{--- (2)}$$

For consistency, C must be divergenceless, $\vec{\nabla}\cdot\vec{C} = 0$ --- (3) because the divergence of a curl is always zero.

If $D(r)$ and $C(r)$ go to zero, sufficiently rapidly at infinity

~~the answer is yes, as I will~~

I claim that

$$\vec{F}(r) = -\vec{\nabla}U + \vec{\nabla}\times\vec{W} \quad \text{--- (4)}$$

where

$$U(r) = \frac{1}{4\pi}\int \frac{D(r')}{\imath}\,d\tau' \quad \text{--- (5)}$$

and

$$W(r) = \frac{1}{4\pi}\int \frac{C(r')}{\imath}\,d\tau' \quad \text{--- (6)}$$

the integrals are over all the space and as always $\imath = |r - r'|$.

$\therefore\ \vec{\nabla}\cdot\vec{F} = -\nabla^2 U = -\frac{1}{4\pi}\int D\nabla^2\left(\frac{1}{r}\right)d\tau' = \int D(r')\,\delta^3(\vec{r}-\vec{r}')\,d\tau' = D(r)$

Again,

$$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = -\nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot W) \quad \text{— (7)}$$

Now, $\vec{\nabla}\cdot\vec{W} = \frac{1}{4\pi}\int \vec{C}(r')\cdot\vec{\nabla}\left(\frac{1}{r}\right)d\tau' = -\frac{1}{4\pi}\int\vec{c}\cdot\vec{\nabla}'\left(\frac{1}{r}\right)d\tau'$ $\left[\because \vec{\nabla} = -\vec{\nabla}'\right.$

$$= \frac{1}{4\pi}\left[\int\frac{1}{r}\vec{\nabla}'\cdot\vec{c}\,d\tau - \oint_s \frac{1}{r}\vec{c}\cdot d\vec{s}\right]$$

Now, from the eqn (3) $\vec{c}$ is divergenless and the surface integral vanishes as long as c goes to zero sufficiently rapidly at infinity. $\vec{\nabla}\cdot\vec{W} = 0$

$$\vec{\nabla}\times\vec{F} = -\nabla^2\vec{W} = -\frac{1}{4\pi}\int c\nabla^2\left(\frac{1}{r}\right)d\tau'$$

$$= \int c(r')\,\delta^3(\vec{r}-\vec{r}')\,d\tau' = c(r)$$

So, the divergence $D(r)$ and the curl $\vec{C}(r)$ of a vector function $F(r)$ are specified, and if both go to zero faster than $1/r^2$ as $r\to\infty$ and if $F(r)$ goes to zero as $r\to\infty$, then F is given uniquely by —

$$F(r) = -\vec{\nabla}U + \vec{\nabla}\times\vec{W}$$

Corollary of Helmholtz theorem :—

Any (differentiable) vector function $F(r)$ that goes to zero faster than $1/r$ as $r\to\infty$ can be expressed as the gradient of a scalar plus the curl of a vector function

$$\vec{F}(r) = \vec{\nabla}\left(\frac{-1}{4\pi}\int\frac{\vec{\nabla}'\cdot\vec{F}(r')}{r}d\tau'\right) + \vec{\nabla}\times\left(\frac{1}{4\pi}\int\frac{\vec{\nabla}'\times\vec{F}(r')}{r}d\tau'\right)$$

Maxwell's eq. free space ($\rho = 0$, $j = 0$)

① $\vec{\nabla}\cdot\vec{E} = 0$ ③ $\vec{\nabla}\cdot\vec{B} = 0$

② $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ ④ $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

taking curl on the both side of eq. ②

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\,\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or, $$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\boxed{\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}} \Rightarrow \text{Wave eq. in 3-D}$$

where, $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ = velocity of em. wave in free space.

● Electromagnetic wave can propagate in vacume.

\## Maxwell displacement current responsible for the existance of the propagation electromagnetic wave.

① $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ ③ $\vec{\nabla}\cdot\vec{B} = 0$

② $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ ④ $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

From eq. ③ $\vec{B} = \vec{\nabla}\times\vec{A}$ —— ⑤

From eq. ④ $\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$

or, $$\vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\therefore \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \quad \text{—— ⑥}$$

Putting the value of $\vec{E}$ and $\vec{B}$ in the eq. ② we get —

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{j} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t}$$

or, $$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right) \quad \text{—— ⑦}$$

Putting $\vec{E}$ and $\vec{B}$ in terms of Potential $\vec{A}$ and ϕ in the eq. ① we get —

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = \rho/\epsilon_0$$

$$\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0 \quad \text{—— ⑧}$$

We have now reduced the set of four Maxwell eqs. of two eqs. But they are still coupled equations.

Gauge Transformations and Gauge Symmetry :

Let us use a transformation from $\vec{A}$ to $\vec{A}'$ such that,

$\vec{A} \rightarrow$ $\quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$ [where Λ be a well behave scalar f^n of space & time coord

Now, $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}' - \vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A}'$

$\therefore$ $\vec{B}$ remain unchange under this transformation

Again,

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}(\vec{A}' - \vec{\nabla}\Lambda)$$

$$\vec{E} = -\vec{\nabla}\left(\phi - \frac{\partial\Lambda}{\partial t}\right) - \frac{\partial\vec{A}'}{\partial t}$$

$\therefore$ $\phi = \phi' + \dfrac{\partial\Lambda}{\partial t}$

$$\vec{E} = -\vec{\nabla}\phi' - \frac{\partial\vec{A}'}{\partial t}$$

Now, $\vec{E}$ is also remain unchange under gauge transformation.

$$\therefore \left\{\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\Lambda \\ \phi' &= \phi - \frac{\partial\Lambda}{\partial t} \end{aligned}\right\} \text{ This is the Gauge transformation.}$$

The invariance of the fields under such gauge transformation is called gauge invariance.

$\boxed{\text{Gauge Symmetry}}$:- Since, both $\vec{E}$ and $\vec{B}$ remain same under these transformation, it is a symmetry transformation for the Maxwell equation. This symetry is called gauge symmetry.

Gauge choice :-

The freedom implied by gauge transformation means that we can choose a set of potentials (A, ϕ) such that

$\boxed{\text{Lorenz Condition}} \Rightarrow$ $\quad \vec{\nabla}\cdot\vec{A} + \dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t} = 0$ —— (9) from eqn (7)

$\Rightarrow$ Gauge choice $\qquad$ where $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$

Now, under gauge choice the eqn (8) and (7) becomes

$$\left\{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\rho/\epsilon_0 \\ \text{and} \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J} \end{aligned}\right. \qquad \text{—— (10)}$$

This will uncouple the pair of equations and leave two inhomogeneous wave equations; one for ϕ and one for A.

To see that, Potentials can always be found to satisfy the Lorentz Condition, suppose that the Potentials $\vec{A}, \phi$ that satisfy eqn. (7) and (8) do not satisfy the eqn. (9).

Then let us make a gauge transformation to Potentials $\vec{A}', \phi'$ and demand that $\vec{A}', \phi'$ satisfy the Lorentz Condition:

$$\therefore \quad \vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0 = \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}$$

$$\cancel{\nabla^2 A} \quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)$$

Now, Preserves the Lorentz Condition, Provided $\vec{A}$ and ϕ satisfy it initially.

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0 \quad \text{—— (11)}$$

Even for Potentials that satisfy the Lorentz Condition there is arbitrariness. Evidently the restricted gauge transformation.

$$\vec{A} \longrightarrow \vec{A}' = A + \vec{\nabla}\Lambda$$

$$\phi \longrightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

where $\quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$

All the Potentials in this restricted class are said to belong to the Lorentz gauge.

\# The Lorentz gauge is Commonly used ~~because~~ -

(1) It leads to the inhomogenious wave equations which treat ϕ and A on equivalent footings.

(2) It is a Concept independent of the Co-ordinate system chosen and so fits naturally into the considerations of special relativity.

Coulomb gauge :- Another usefull gauge for the Potentials is the Coulomb gauge or transverse gauge.

This is the gauge in which $\vec{\nabla}\cdot\vec{A} = 0$

Now from the eqn. (8) we get $\quad \nabla^2\phi = -\rho/\epsilon_0$

and soln. is $\quad \phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)\, dv'}{|\vec{r}-\vec{r}'|}$

Plasma Physics

Plasma :- A plasma is an ionized gas consisting of charged particles (e.g., electrons and ions).

At very high temperature (50,000K) collisions between gas particle causes the ionization of the gas.

⊛ "Plasma is often mentioned as the fourth state of matter" — Explain.

⇒ A highly ionized gas in which the number of free electrons are approximately equal to the number of positive ions, some times described as fourth state of matter, plasma occur in interstellar space, in the atmospheres of star (including sun), in discharge tubes and in experimental thermo-nuclear reactors.

Because the particles in a plasma are charged, its behaviour differs in some respects from that of gas. Plasmas can be created in the laboratory by heating, a low pressure gas until the mean kinetic energy of the gas particles is comparable to the ionization potential of the gas ~~Particle~~ atom or molecule.

$\nu_c << \nu$ → plasma oscillation

where, ν_c ⇒ Collision frequency

$\nu_c > \nu$ ⇒ Magneto-hydrodynamics

The plasma column will move as a whole, the effecti charge density will be zero.

But, & since ~~here~~ there are a large no. of carrier the conductivity will be very high ($\sigma \to \infty$).

Propagation of electromagnetic waves in a dilute Plasma.

⇒ Electric current is carried by electrons and ionized atoms. Because the electrons are much less massive than the ions, the current is dominated by the electron motion.

$\left(a_e = F/m_e \gg F/m_{ion} = a_{ion}\right)$

use the classical electron model for the current.

the free electrons in a plasma obey the equation of motion —

$$m\frac{d^2\vec{r}}{dt^2} = -\gamma\frac{d\vec{r}}{dt} - e\vec{E_0}e^{-i\omega t}$$

electric force

damping factor.

the steady-state solution is —

$$\vec{r}(t) = \frac{e}{m\omega^2 + i\omega\gamma}\vec{E_0}e^{-i\omega t}$$

the electron position oscillates at the driving frequency.

and the velocity is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{-i\omega e}{m\omega^2 + i\omega\gamma}\vec{E}(t)$$

Current density

Let n_e = electron density (# electron/m^3)

Then $\vec{J} = -e\vec{v}\, n_e$ unit: $C \cdot \frac{m}{s \cdot m^3} = A/m^2$

So, $\vec{J} = \sigma \vec{E}$ (like ohm's law)

where, $\sigma(E) = \frac{i\omega e^2 n_e}{m\omega^2 + i\omega\gamma}$ i.e the Conductivity is Complex.

Case of a Dilute Plasma

It's not simply that n_e is small, because σ is proportional to n_e.

What we mean is that Collisions are rare.

Then γ is small, i.e $i\omega\gamma << m\omega^2$,

So, we'll take $\sigma(\omega) = \frac{ie^2 n_e}{m\omega}$

Purely imaginary σ

$\boxed{\text{there is no energy loss}}$

Now, $\frac{dP}{dV} = \vec{J}\cdot\vec{E}$ ⇒ Power transfer from the electromagnetic field to the electrons.

$\vec{E} = \mathrm{Re}\, \vec{E_0}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$= \vec{E_0} \cos(\vec{k}\cdot\vec{x} - \omega t)$

$\vec{J} = \mathrm{Re}(\sigma\vec{E}) = -\frac{e^2 n_e}{m\omega}\vec{E_0} \sin(\vec{k}\cdot x - \omega t)$

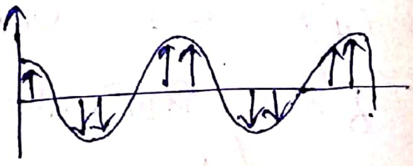
∴ the electric field and current oscillates 90° degree out of phase;

Now, $\vec{E}\cdot\vec{J} = -\frac{e^2 \eta_e}{m\omega} E_0^2 \underbrace{\cos(kx-\omega t)\sin(kx-\omega t)}_{\text{average value is 0}}$

The Energy sloshes back and forth between the field and the current, but there is no energy loss.

Plasma frequency & Dispersion relation

the Dispersion Relation



$$\vec{E} = E_0 \hat{j}\, e^{i(kx-\omega t)}$$

$$\vec{B} = \frac{k}{\omega} E_0 \hat{k}\, e^{i(kx-\omega t)}$$

$$\underbrace{\vec{\nabla}\times\vec{B}}_{} = \frac{kE_0}{\omega}\, ik\, \hat{i}\times\hat{k}\, e^{i(kx-\omega t)} = \frac{-ik^2}{\omega} E_0 \hat{j}\, e^{i(kx-\omega t)}$$

$$\therefore \hookrightarrow \mu_0\sigma\vec{E} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = (\mu_0\sigma - \mu_0\epsilon_0 i\omega) E_0\hat{j}\, e^{i(kx-\omega t)}$$

$$\therefore \frac{-ik^2}{\omega} = -\frac{i\omega}{c^2} + \mu_0\frac{ie^2\eta_e}{m\omega}$$

$$\therefore \boxed{k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}} \quad \text{where } \omega_p^2 = \frac{e^2\eta_e}{m\epsilon_0}$$

$\Downarrow$

Dispersion relation i.e the relation betn k and ω,

and Plasma frequency, $\omega_p = \sqrt{\frac{e^2\eta_e}{m\epsilon_0}}$

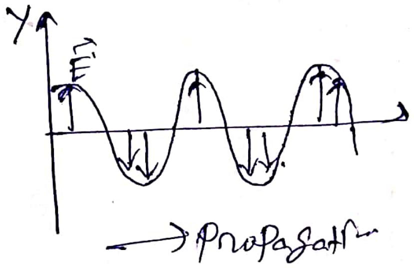
Propagation of Electro-magnetic Waves

(I) High frequency range –

High frequency, $\vec{E} = E_0 \hat{j} e^{i(kx - \omega t)}$ (real part)

If $\omega > \omega_p$ then k is real and the wave propagates with const amplitude.

$$\vec{E} = \vec{E_0} \hat{j} \cos(kx - \omega t)$$



Dispersion, Phase velocity is

$$V_{phase} = \frac{\omega}{k}$$

$$V_{phase} = \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2 - \omega_p^2}}$$ depends on frequency

(II) Low frequency :-

$$k = \frac{\sqrt{\omega_p^2 - \omega^2}}{c}$$

If $\omega < \omega_p$ the k is purely imaginary;

$k = iK$ where K is real $\Rightarrow$ attenuation
no propagation.

$$\vec{E} = E_0 \hat{j} e^{i(iKx - \omega t)} = E_0 \hat{j} e^{-Kx} e^{-i\omega t}$$

$$\text{real Part} = E_0 \hat{j} e^{-Kx} \cos \omega t$$

In electrostatics an electric field does not penetrate into a conductor (e.g. a plasma); the electrons move to screen the electric field.

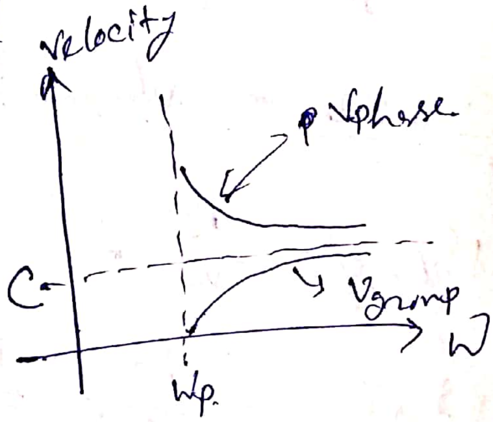
Exercise : For low frequency ($\omega < \omega_p$) Calculate the attenuation length.

$$V_{phase} = \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2-\omega_p^2}} > c \qquad \boxed{k^2 = \frac{\omega^2-\omega_p^2}{c^2}}$$

$$V_{group} = \frac{d\omega}{dk} = \frac{\sqrt{\omega^2-\omega_p^2}}{\omega}\, c < c \Leftarrow \text{i.e } c^2/V_{phase}$$

$$dk^2 = 2k\,dk = \frac{d\omega^2}{c^2} = \frac{2\omega\, d\omega}{c^2}, \quad \frac{d\omega}{dk} = \frac{c^2 k}{\omega}$$

$$\boxed{V_{phase} \times V_{group} = c^2}$$



i.e Propagation of finite electromagnetic waves, i.e waves with finite longitudinal ~~extent~~ extent (as opposed to ideal plane waves) (pulses or wave packets).

Skin depth in plasma:-

$$\vec{E} = \vec{E_0}\, e^{i(k\cdot x - \omega t)}$$

From dispersion relation, we obtain, $k = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$

- If $\omega > \omega_p$ (like vacuum) k is real, $\vec{E} = \vec{E_0}\, e^{i(kx-\omega t)}$, wave propagates without decay.
- If $\omega < \omega_p$ (like conductor) $k = i\frac{1}{c}\sqrt{\omega_p^2-\omega^2} = i|k|$ is imaginary.
  $\vec{E} = \vec{E_0}\, e^{i\omega t}\, e^{-|k|x} \Rightarrow$ wave decays in ~~x~~ direction and will be reflected.

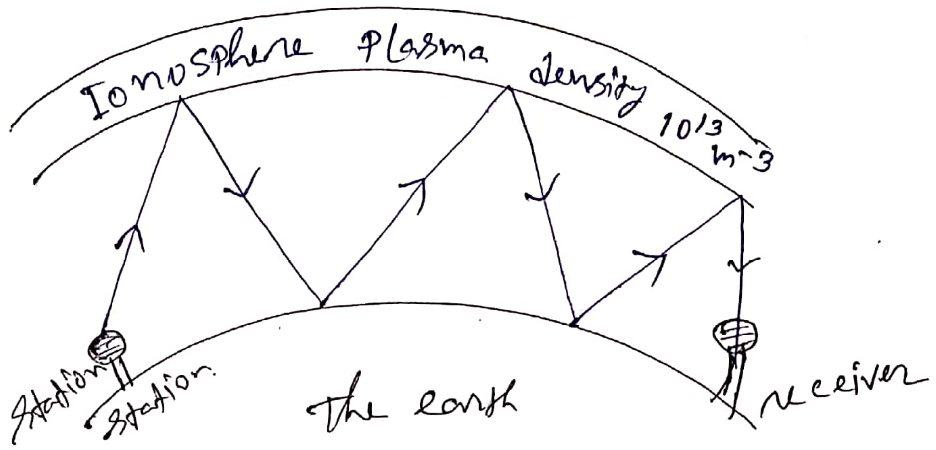
If $\omega \ll \omega_p$, $k \simeq i\frac{\omega_p}{c} = i\frac{1}{\delta}$, $\boxed{\delta = \frac{c}{\omega_p}} \Rightarrow$ Skin depth in pla

Application to Propagation through ionosphere.

In Ionospheric Plasma (hight 50 km to 100 km) has a typical density of $n_e = 10^{13}/m^3$

$$f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{e^2 n_e}{m\epsilon_0}} \simeq 28 \text{ MHz}.$$

Short wave radio ($f \sim 10$ MHz) relies on the multiple reflection between the ionospheric plasma layer and the earth to reach a distant receiver.



Earth is "Conductor" ($\sigma \sim 10^{-2}$ S/m) as long as the impedance $|Z| = \left|\sqrt{\frac{-i\omega\mu_0}{\sigma}}\right| \ll Z_{air} = \sqrt{\frac{\mu_0}{\epsilon_0}}$

which requires $\omega \ll \frac{\sigma}{\epsilon_0}$

or, $f \ll \frac{\sigma}{2\pi\epsilon_0} = \frac{10^{-2}}{2\pi \times 8.85 \times 10^{-12}} = 180$ MHz

For $f = 10$ MHz, the earth is good Conductor.